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Damped finite-time singularity driven by noise

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We consider the combined influence of linear damping and noise on a dynamical finite-time singularity model for a single degree of freedom. We find that the noise effectively resolves the finite-time singularity and replaces it by a first-passage-time distribution or absorbing state distribution with a peak at the singularity and a long time tail. The damping introduces a characteristic cross-over time. In the early time regime the probability distribution and first-passage-time distribution show a power law behavior with scaling exponent depending on the ratio of the nonlinear coupling strength to the noise strength. In the late time regime the behavior is controlled by the damping. The study might be of relevance in the context of hydrodynamics on a nanometer scale, in material physics, and in biophysics.

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I. INTRODUCTION

The influence of noise on the behavior of nonlinear dynamical system is a recurrent theme in modern statistical physics [1]. In a particular class of systems the nonlinear character gives rise to finite-time singularities, that is, solutions, which cease to be valid beyond a particular finite time span. One encounters finite-time singularities in stellar structure, turbulent flow, and bacterial growth [2–4]. The phenomenon is also seen in Euler flows and in free-surface flows [5–8]. Finally, finite-time singularities are encountered in modeling in econophysics, geophysics, and material physics [9–13].

In the context of hydrodynamical flow on a nanoscale [14], where microscopic degrees of freedom come into play, it is a relevant issue so as to how noise influences the hydrodynamical behavior near a finite-time singularity. Leaving aside the issue of the detailed reduction of the hydrodynamical equations to a nanoscale and the influence of noise on this scale to further study, we assume in the present context that a single variable or ''reaction coordinate'' effectively captures the interplay between the singularity and the noise.

Generally an equation of motion for a single degree of freedom x, describing a dynamical phenomenon with damping and imposed noise, is second order in time and has the form

$$m\frac{d^2x}{dt^2}+\Gamma\frac{dx}{dt}=F(x)+\eta. \tag{1.1}$$

Here m is the mass, Γ is the damping constant, η is the imposed noise, and $F(x)\propto 1/x^{1+\mu}$ is a singular force generating the finite-time singularity.

In the overdamped or high friction limit with Γ large we can neglect the inertial second order term thus, we obtain, subject to a rescaling of time, the nonlinear Langevin equation

$$\frac{dx}{dt}=-\frac{\lambda}{2|x|^{1+\mu}}+\eta, \qquad \langle\eta\eta\rangle(t)=\Delta\delta(t), \tag{1.2}$$

which was studied in detail in a recent paper [15].

The model is characterized by the coupling parameter λ determining the amplitude of the singular term, the index $\mu\geqslant 0$ characterizing the nature of the singularity, and the noise parameter Δ determining the strength of the noise correlations. Specifically, in the case of a thermal environment at temperature T the noise strength $\Delta\propto T$.

In the absence of noise this model exhibits a finite-time singularity at a time t_0, where the variable x vanishes with a power law behavior determined by μ. When noise is added, the finite-time singularity event at t_0 becomes a statistical event and is conveniently characterized by a first-passage-time distribution $W(t)$ [16]. For vanishing noise we have $W(t)=\delta(t-t_0)$, restating the presence of the finite-time singularity. In the presence of noise, $W(t)$ develops a peak about $t=t_0$, vanishes at short times, and acquires a long time tail.

The model in Eq. (1.2) has also been studied in the context of persistence distributions related to the nonequilibrium critical dynamics of the two-dimensional XY model [17] and in the context of non-Gaussian Markov processes [18]. Finally, regularized for small x, the model enters in connection with an analysis of long-range correlated stationary processes [19].

From our analysis in Ref. [15] it followed that for $\mu=0$, the logarithmic case, the distribution at long times is given by the power law behavior

$$W(t)\sim t^{-\alpha}, \quad \alpha=\frac{3}{2}+\frac{\lambda}{2\Delta}. \tag{1.3}$$

For vanishing nonlinearity, i.e., $\lambda=0$, the finite-time singularity is absent and the Langevin equation (1.2) describes a simple random walk of the reaction coordinate, yielding the well-known exponent $\alpha=3/2$ [16,20,21]. In the nonlinear case with a finite-time singularity the exponent attains a nonuniversal correction, depending on the ratio of the nonlinear

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strength to the strength of the noise; for a thermal environment the correction is proportional to $1/T$. In the generic case for $\mu>0$, we found that the falloff is slower and that the correction to the random walk result is given by a stretched exponential

$$W(t)\sim t^{-3/2}\exp[-A(t^{-\mu/(2+\mu)}-1)], \tag{1.4}$$

where $A\to\lambda/\Delta\mu$ for $\mu\to 0$.

Although the model system described by Eq. (1.2) can be conceived to originate from the overdamped limit of an equation of motion with a singular force term, it is a valid issue whether additional damping, for example, a linear damping term, can influence the character of the noise resolution of the finite-time singularity, in particular the long time behavior of the first-passage-time distribution $W(t)$.

One way of providing a physical motivation is to consider a first order equation of motion for the velocity v with a singular force depending on the velocity, yielding the finite-time singularity

$$m\frac{dv}{dt}+\Gamma v=F(v)+\eta, \tag{1.5}$$

alternatively, we refer to the general time-dependent Ginzburg-Landau scheme [22,23] in the context of dynamical critical phenomena and pattern formation exemplified by the Langevin equation

$$\frac{\partial\phi}{\partial t}=-\Gamma\frac{\delta F}{\delta\phi}+R(\phi)+\eta. \tag{1.6}$$

Assuming a single degree of freedom $\phi=x$, a free energy of the form $F\propto x^2$, and a "mode coupling term" R yielding the singular force, this equation also gives rise to a linear damping term.

In the present paper we consider the case of additional linear damping and thus proceed to extend the analysis in Ref. [15]. Here we shall only consider the logarithmic case for $\mu=0$:

$$\frac{dx}{dt}=-\gamma x-\frac{\lambda}{2|x|}+\eta, \quad \langle\eta\eta\rangle=\Delta\delta(t). \tag{1.7}$$

In addition to the coupling parameter λ and the noise parameter Δ, this model is also characterized by the damping constant γ. Assuming for convenience a dimensionless variable x, the coupling and the noise strengths λ and Δ have the dimension 1/time. The ratios λ/Δ and γ/Δ are thus dimensionless parameters characterizing the behavior of the system.

It follows from our analysis below that the damping constant sets an inverse time scale $1/\gamma$. At intermediate time scales for $\gamma t\ll 1$ the distribution exhibits the same power law behavior as in the undamped case given by Eq. (1.3). At long times for $\gamma t\gg 1$, the distribution falls off exponentially with time constant $1/\gamma(1+\lambda/\Delta)$, i.e.,

$$W(t)\propto\exp[-\gamma(1+\lambda/\Delta)t]. \tag{1.8}$$

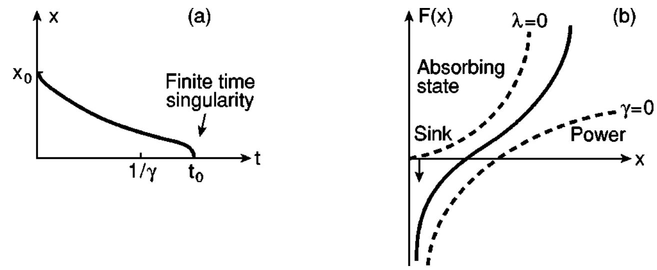

FIG. 1. In (a) we show the time evolution of the single degree of freedom x. At times shorter than the cross-over time $1/\gamma$ the variable x falls off exponentially. At times beyond $1/\gamma$ the variable x reaches the absorbing state $x=0$ at a finite time t_0. In (b) we depict the free energy $F(x)$ driving the equation. For $\lambda=0$, the free energy forms a confining harmonic well, for $\gamma=0$ we have the absorbing state case discussed in Ref. [15]. In the general case the absorbing state $x=0$ corresponds to the sink in $F(x)$.

The paper is organized in the following manner. In Sec. II we introduce the finite-time singularity model with linear damping and discuss its properties. In Sec. III we review the weak noise WKB phase space approach to the Fokker-Planck equation, apply it to the finite-time singularity problem with damping, and discuss the associated dynamical phase space problem and the long time properties of the distributions. In Sec. IV we derive an exact solution of the Fokker-Planck equation and present an expression for the first-passage-time distribution. In Sec. V we present a summary and a conclusion. In the present treatment we draw heavily on the analysis in Ref. [15]. In Appendix A, aspects of the exact solution are discussed in more detail; in Appendix B, we consider the weak noise limit of the exact solution.

II. MODEL

In terms of a free energy or potential F we can express Eq. (1.7) in the form

$$\frac{dx}{dt}=-\frac{1}{2}\frac{dF}{dx}+\eta(t), \tag{2.1}$$

where F has the form

$$F=\gamma x^2+\lambda\ln|x|. \tag{2.2}$$

The free energy has a logarithmic sink and drives x to the absorbing state $x=0$. For large x, the free energy has the form of a harmonic well potential confining the motion. In Fig. 1 we have depicted the free energy in the various cases.

A. The noiseless case

In the case of vanishing noise Eq. (2.1) is readily solved. We obtain

$$x(t)=\left[\frac{\lambda}{2\gamma}\right]^{1/2}[e^{2\gamma(t_0-t)}-1]^{1/2}, \tag{2.3}$$

with a finite-time singularity at

$$t_0=\frac{1}{2\gamma}\ln\left|1+\frac{2\gamma}{\lambda}x_0^2\right|. \tag{2.4}$$

The initial value of x is x_0 at time $t=0$. In the presence of damping, x initially falls off exponentially due to the confining harmonic potential with a time constant $1/\gamma$. For times beyond $1/\gamma$, the nonlinear term takes over and drives x to zero at time t_0, i.e., x falls into the sink in F. In Fig. 1 we have shown the noiseless solution $x(t)$.

B. The noisy case

Summarizing the discussion in Ref. [15], the stochastic aspects of the finite-time singularity in the presence of noise are analyzed by focusing on the time-dependent probability distribution $P(x,t)$ and the derived first-passage-time distribution or absorbing state probability distribution $W(t)$. The distribution $P(x,t)$ is defined according to [21,24] $P(y,t)=\langle\delta(y-x(t))\rangle$ where x is a stochastic solution of Eq. (2.1) and $\langle\cdots\rangle$ indicates an average over the noise η driving x. In the absence of noise $P(y,t)=\delta[y-x(t)]$, where x is the deterministic solution given by Eq. (2.3) and depicted in Fig. 1. At time $t=0$ the variable x evolves from the initial condition x_0 implying the boundary condition $P(x,0)=\delta(x-x_0)$.

At short times x is close to x_0, and the singular term and the damping term are not yet operational. In this regime we obtain ordinary random walk with the Gaussian distribution $P(x,t)=(2\pi\Delta t)^{-1/2}\exp[-(x-x_0)^2/2\Delta t]$. At a time scale, given by $1/\gamma$, the damping drives x towards a stationary distribution, given by $P\propto\exp(-F/\Delta)$. However, at longer times beyond the scale $1/\gamma$, the barrier $\lambda/2x$ comes into play preventing x from crossing the absorbing state $x=0$. This is, however, a random event which can occur at an arbitrary time instant, i.e., the finite-time singularity, at t_0 in the deterministic case, is effectively resolved in the noisy case. For not too large noise strength the distribution is peaked about the noiseless solution and vanishes for $x\rightarrow 0$, corresponding to the absorbing state, implying the boundary condition

$$P(0,t)=0. \tag{2.5}$$

In order to model a possible experimental situation the first-passage-time distribution or absorbing state distribution $W(t)$ is of more direct interest [24,25].

Since $P(0,t)=0$ for all t due to the absorbing state, the probability that x is not reaching $x=0$ in time t is thus given by $\int_0^\infty P(x,t)dx$, implying that the probability $-dW$ that x does reach $x=0$ in time t is $-dW=-\int_0^\infty dx dt(dP/dt)$, yielding the absorbing state distribution $W(t)=-\int_0^\infty dx\ \partial P(x,t)/\partial t$ [21]. In the absence of noise $P(x,t)=\delta[x-x(t)]$ and $W(t)=\delta(t-t_0)$, in accordance with the finite-time singularity at $t=t_0$. For weak noise $W(t)$ peaks about t_0 with vanishing tails for small t and large t.

The distribution $P(x,t)$ satisfies the Fokker-Planck equation [24,25]

$$\frac{\partial P}{\partial t}=\frac{1}{2}\frac{\partial}{\partial x}\left[\frac{dF}{dx}P+\Delta\frac{\partial P}{\partial x}\right], \tag{2.6}$$

in the present case subject to the boundary conditions $P(x,0)=\delta(x-x_0)$ and $P(0,t)=0$. The Fokker-Planck equation has the form of a conservation law $\partial P/\partial t+\partial J/\partial x=0$ defining the probability current $J=(1/2)(dF/dx)P-(1/2)\Delta\partial P/\partial x$. For $W(t)$ we obtain the expression

$$W(t)=\frac{1}{2}\left[\frac{dF}{dx}P+\Delta\frac{\partial P}{\partial x}\right]_{x=0}, \tag{2.7}$$

to be used in our further analysis. Note that there is a sign error in Eq. (3.8) in Ref. [15].

III. WEAK NOISE APPROACH

In this section we apply a weak noise canonical phase space approach to the damped finite-time singularity model and infer the general long time behavior. The treatment follows closely the analysis in Ref. [15].

A. The phase space method

From a structural point of view the Fokker-Planck equation (2.6) has the form of an imaginary-time Schrödinger equation $\Delta\partial P/\partial t=HP$, driven by the Hamiltonian or Liouvillian H. The noise strength Δ plays the role of an effective Planck constant and P corresponds to the wave function. Drawing on this parallel we have in recent work in the context of the Kardar-Parisi-Zhang equation for a growing interface elaborated on a weak noise nonperturbative WKB phase space approach to a generic Fokker-Planck equation for extended system [26–28]. In the case of a single degree of freedom this method amounts to the eikonal approximation [21,25,29], see also Refs. [30,31]. For systems with many degrees of freedom the method has, for example, been expounded in Ref. [32], based on the functional formulation of the Langevin equation [33,34]. In the present formulation [26–28] the emphasis is on the canonical phase space analysis and the use of dynamical system theory [35,36].

The weak noise WKB approximation corresponds to the ansatz $P\propto\exp[-S/\Delta]$. The weight function or action S then to leading asymptotic order in Δ satisfies a Hamilton-Jacobi equation $\partial S/\partial t+H=0$, which in turn implies a *principle of least action* and Hamiltonian equations of motion [37,38]. In the present context the Hamiltonian takes the form

$$H=\frac{1}{2}p\left(p-\frac{\lambda}{x}-2\gamma x\right), \tag{3.1}$$

yielding the Hamilton equations of motion

$$\frac{dx}{dt}=-\gamma x-\frac{\lambda}{2x}+p, \tag{3.2}$$

$$\frac{dp}{dt}=\gamma p-\frac{1}{2}\frac{\lambda}{x^2}p. \tag{3.3}$$

These equations replace the Langevin equation (1.7) with the noise η represented by the momentum $p=\partial S/\partial x$, conjugate to x. Equations (3.2) and (3.3) determine orbits in a canonical

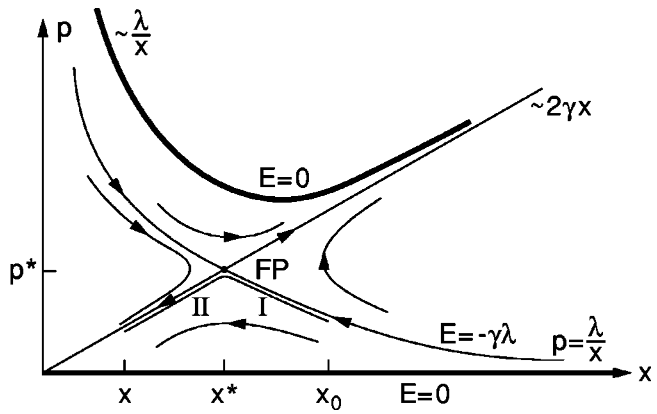

FIG. 2. We show the topology of phase space. The bold lines indicate the zero energy submanifolds. The invariant heteroclinic orbits $p=2\gamma x$ and $p=\lambda/x$, passing through the saddle point FP at $(x^*,p^*)=[(\lambda/2\gamma)1/2,(2\gamma\lambda)^{1/2}]$, have energy $-\gamma\lambda$. The arrows indicate the direction of the flow. The long time orbit from x_0 to x passes close to the fixed point. The part of the orbit following the invariant manifold $p=\lambda/x$ and entering in our long time estimate is denoted by I; the part of the orbit close to the $p=2\gamma x$ manifold is denoted by II.

phase space spanned by x and p. Since the system is conserved the orbits lie on the constant energy manifold(s) given by $E=H$. The action associated with an orbit from x_0 to x in time t has the form

$$S(x_0\rightarrow x,t)=\int_0^t dt\left[p\frac{dx}{dt}-H\right]. \tag{3.4}$$

According to the ansatz the probability distribution is then given by

$$P(x,t)=P(x_0\rightarrow x,t)\propto\exp\left[-\frac{S(x_0\rightarrow x,t)}{\Delta}\right]. \tag{3.5}$$

B. Long time orbits

The zero-energy manifolds delimiting the phase space orbits follow from Eq. (3.1) and are given by $p=0$ and $p=2\gamma x+\lambda/2x$. The $p=0$ submanifold corresponds to the noiseless or deterministic case discussed above. The $p=\lambda/x+2\gamma x$ submanifold corresponds to the noisy case. By insertion in Eq. (3.3) we obtain $dx/dt=\gamma x+\lambda/2x$, i.e., the motion on the noisy submanifold is time reversed of the motion on the noiseless submanifold. The orbit structure in phase space is moreover controlled by the hyperbolic fixed point at $(x^*,p^*)=[(\lambda/2\gamma)^{1/2},(2\gamma\lambda)^{1/2}]$. The heteroclinic orbits passing through the fixed point are given by $p=\lambda/x$ and $p=2\gamma x$, and the energy of the invariant manifold is $E^*=-\gamma\lambda$. In Fig. 2 we have depicted the phase space with the zero-energy manifolds, the fixed point, the heteroclinic orbits, and some characteristic orbits.

The long time behavior of the distribution is determined by an orbit from x_0 to x traversed in time t. In the long time limit this orbit must pass close to the hyperbolic fixed point. Note that in the limit $\gamma\rightarrow 0$, the fixed point migrates to infinity in the x direction and the long time orbits approach the zero-energy submanifolds which thus determine the asymptotic properties as discussed in Ref. [15].

Independent of whether the initial value x_0 is greater or smaller than the fixed point value x^*, the long time orbit follows the invariant $p=\lambda/x$ manifold towards the fixed point. At the fixed point the orbit slows down and then speeds up again as the orbit follows the other invariant manifold $p=2\gamma x$ towards the endpoint x reached in time t. This behavior is also depicted in Fig. 2.

This scenario allows a simple analysis of the long time behavior of the distribution $P(x,t)$ and the first-passage-time distribution $W(t)$. Close to the invariant manifolds with energy $E^*=-\gamma\lambda$, the action associated with an orbit from x_0 to x follows from Eq. (3.4) and is given by

$$S=-E^*t+\int_{x_0}^{x^*}dx\frac{\lambda}{x}+\int_{x^*}^{x}dx2\gamma x, \tag{3.6}$$

or, denoting the relevant manifolds by a subscript, see Fig. 2,

$$S=\lambda\gamma t+\lambda\ln\left|\frac{x^*}{x_0}\right|_{\mathrm{I}}+\gamma(x^2-x^{*2})_{\mathrm{II}}. \tag{3.7}$$

At long times we only have to consider the contribution from the orbit leading up to the fixed point. Inserting the manifold condition $p=\lambda/x$ in the equation of motion (3.2) we thus obtain $dx/dt=-\gamma x+\lambda/2x$ with solution

$$x(t)^2=x_0^2e^{-2\gamma t}+x^{*2}(1-e^{-2\gamma t}). \tag{3.8}$$

C. Discussion

It follows from Eq. (3.8) that the damping γ sets an inverse time scale delimiting two kinds of characteristic behavior. First, for $t\rightarrow\infty$ the orbit approaches the fixed point x^*. For $\gamma t\gg 1$ we have $x^2=x^{*2}[1-\exp(-2\gamma t)]$ and x approaches the fixed point in an exponential fashion. On the other hand, in the intermediate time region for $\gamma t\ll 1$ and for $\gamma t\gg x_0^2$ and $\lambda t\gg x_0^2$ we obtain $x^2=x_0^2+2t\gamma x^*=x_0^2+\lambda t\sim\lambda t$.

By insertion in the expression (3.7) for the action we then obtain in the late time regime for $\gamma t\gg 1$

$$S(t)\sim\lambda\gamma t-\frac{\lambda}{2}e^{-2\gamma t}, \tag{3.9}$$

yielding the distribution and ensuing first-passage-time distribution

$$P(t)\propto W(t)\propto\exp(-\lambda\gamma t/\Delta). \tag{3.10}$$

Likewise, we have in the intermediate time regime $\gamma t\ll 1$

$$S(t)\sim\lambda\gamma t+\frac{\lambda}{2}\ln|t|, \tag{3.11}$$

giving rise to the distribution and first-passage-time distribution

$$P(t) \propto W(t) \propto |t|^{-(\lambda/2\Delta)}. \tag{3.12}$$

These results hold in the weak noise limit. We note that at long times, $W(t)$ falls off exponentially with a time constant given by $\Delta/\lambda\gamma$. In the intermediate time regime, $W(t)$ exhibits a power law behavior with exponent $-\lambda/2\Delta$, independent of $1/\gamma$ defining the cross-over time. These results will also be recovered from the exact solution discussed in the following sections.

IV. EXACT SOLUTION

In this section we return to the Fokker-Planck equation (2.6) and present an exact solution. This solution is an extension of the solution presented in Ref. [15] and the analysis proceeds in much the same way. Details are discussed in Appendixes A and B.

Quantum particle in a harmonic potential with centrifugal barrier

The Fokker-Planck equation has the form

$$\frac{\partial P}{\partial t} = \frac{\Delta}{2}\frac{\partial^2 P}{\partial x^2} + \left(\gamma x + \frac{\lambda}{2x}\right)\frac{\partial P}{\partial x} + \left(\gamma - \frac{\lambda}{2x^2}\right)P. \tag{4.1}$$

Eliminating the first order term by means of the gauge transformation

$$\exp(h) = |x|^{-\lambda/2\Delta} e^{-\gamma x^2/2\Delta}, \tag{4.2}$$

we can express the equation in the form

$$-\Delta \frac{\partial}{\partial t}[\exp(-h)P] = H[\exp(-h)P], \tag{4.3}$$

where the Hamiltonian H driving P is given by

$$H = -\frac{1}{2}\Delta^2 \frac{\partial^2}{\partial x^2} + \frac{\lambda^2}{8}\left(1 + \frac{2\Delta}{\lambda}\right)\frac{1}{x^2} + \frac{\Delta\gamma}{2}\left(\frac{\lambda}{\Delta} - 1\right) + \frac{\gamma^2}{2}x^2. \tag{4.4}$$

This Hamiltonian describes the motion of a unit mass quantum particle in one dimension in a harmonic potential subject to a centrifugal barrier of strength $(\lambda^2/8)(1+2\Delta/\lambda)$ at the origin; Δ plays the role of an effective Planck constant. Note that in Eq. (6.4) in Ref. [15] the factor $\Delta/2$ should read $1/\Delta$.

For $\lambda=0$ and $\gamma=0$, both the barrier and the confining potential are absent; the spectrum of H forms a band and the particle can move over the whole axis. This case corresponds to ordinary random walk [21]. Incorporating the absorbing state condition in Eq. (2.5) by means of the method of mirrors we obtain the results presented in Ref. [15], i.e.,

$$P(x,t) = (2\pi\Delta t)^{-1/2}\left(\exp\left[-\frac{(x-x_0)^2}{2\Delta t}\right] - \exp\left[-\frac{(x+x_0)^2}{2\Delta t}\right]\right), \tag{4.5}$$

in the half space $x \geq 0$, and for the absorbing state distribution

$$W(t) = \left(\frac{2}{\pi}\right)^{1/2} x_0 \exp(-x_0^2/2\Delta t)(\Delta t)^{-3/2}. \tag{4.6}$$

For $\lambda \neq 0$ and $\gamma=0$ the particle cannot cross the barrier and is confined to either half space; this corresponds to the case of a finite-time singularity subject to noise and an absorbing state at $x=0$, and was discussed in detail in Ref. [15]; for reference we give the obtained results below. Note that x and x_0 should be interchanged in Eq. (6.5) and that a factor Δ is missing in Eq. (6.6) in Ref. [15],

$$P(x,t) = \frac{x_0^{(\lambda/2\Delta)+(1/2)}}{x^{(\lambda/2\Delta)-(1/2)}} \frac{\exp\left(-\dfrac{x^2+x_0^2}{2\Delta t}\right)}{\Delta t} I_{(1/2)+(\lambda/2\Delta)}\left(\frac{x x_0}{\Delta t}\right). \tag{4.7}$$

$$W(t) = \frac{2\Delta x_0^{1+\lambda/\Delta}}{\Gamma[(1/2)+(\lambda/2\Delta)]} \times \exp(-x_0^2/2\Delta t)(2\Delta t)^{-(3/2)-(\lambda/2\Delta)}. \tag{4.8}$$

In the present case for $\lambda \neq 0$ and $\gamma \neq 0$ the problem corresponds to the motion of a particle in a harmonic potential with a centrifugal barrier at $x=0$. The spectrum is discrete and becomes continuous for $\gamma=0$. As discussed in detail in Appendix A the problem is readily analyzed in terms of confluent hypergeometric functions, more specifically Laguerre polynomials [39,40]. Incorporating the initial condition $P(x,0)=\delta(x-x_0)$ and introducing the time scaled variables

$$x = \tilde{x}\exp(-\gamma t/2), \tag{4.9}$$

$$x_0 = \tilde{x}_0 \exp(+\gamma t/2), \tag{4.10}$$

we find for $P(x,t)$, see Appendix A,

$$P(x,t) = \frac{\tilde{x}_0^{(\lambda/2\Delta)+(1/2)}}{\tilde{x}^{(\lambda/2\Delta)-(1/2)}} \frac{\gamma e^{\gamma t/2}}{\Delta \sinh\gamma t} \exp\left[-\frac{\gamma(\tilde{x}^2+\tilde{x}_0^2)}{2\Delta\sinh\gamma t}\right] \times I_{(1/2)+(\lambda/2\Delta)}\left(\frac{\gamma}{\Delta}\frac{\tilde{x}\tilde{x}_0}{\sinh\gamma t}\right), \tag{4.11}$$

and correspondingly for the absorbing state distribution

$$W(t) = \frac{2\Delta\tilde{x}_0^{1+\lambda/\Delta}}{\Gamma[(1/2)+(\lambda/2\Delta)]}\exp\left[-\frac{\gamma\tilde{x}_0^2}{2\Delta\sinh\gamma t}\right] \times \exp(\gamma t)\left(\frac{\gamma}{2\Delta\sinh\gamma t}\right)^{(3/2)+(\lambda/2\Delta)}. \tag{4.12}$$

In Eqs. (4.7) and (4.11), I_ν is the Bessel function of imaginary argument, $I_\nu(z)=(-i)^\nu J_\nu(iz)$ [41].

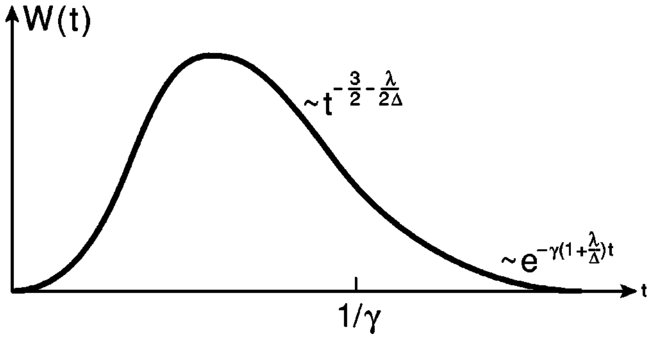

FIG. 3. We sketch the first-passage-time distribution $W(t)$ as a function of t. In the limit $t\rightarrow 0$, $W(t)$ vanishes exponentially; about the finite-time singularity $W(t)$ exhibits a maximum. At intermediate times for $\gamma t \ll 1$ the distribution exhibits a power law behavior with scaling exponent $(3/2)+(\lambda/2\Delta)$. In the long time limit for $\gamma t \gg 1$ an exponential falloff with time constant $\gamma(1+\lambda/\Delta)$ characterizes the behavior of $W(t)$.

For consistency we have in Appendix B analyzed the weak noise limit $\Delta\rightarrow 0$ of the exact solution in Eq. (4.11) and shown that the trajectory converges to the noiseless orbit given by Eq. (2.3).

V. DISCUSSION AND CONCLUSION

Focusing on the expression (4.12) for the first-passage-time distribution $W(t)$ we note that the damping constant γ defines two distinct time regimes, where $1/\gamma$ sets the characteristic crossover time. In the long time limit for $\gamma t \gg 1$ the damping constant controls the behavior of $W(t)$. From Eq. (4.12) we infer

$$W(t) \propto \frac{2\Delta x_0^{1+\lambda/\Delta}}{\Gamma[(1/2)+(\lambda/2\Delta)]}\left(\frac{\gamma}{\Delta}\right)^{(3/2)+(\lambda/2\Delta)} \times \exp[-\gamma(1+\lambda/\Delta)t], \tag{5.1}$$

i.e., $W(t)$ falls off exponentially with an effective damping constant $\gamma[1+(\lambda/\Delta)]$ renormalized by the ratio λ/Δ of the nonlinear strength to the noise strength. We note that for $\Delta\rightarrow 0$ the result is in accordance with the weak noise phase space derivation in Sec. III. In the intermediate time regime for $\gamma t \ll 1$ the damping constant γ drops out and we obtain

$$W(t) \propto \frac{2\Delta x_0^{1+\lambda/\Delta}}{\Gamma[(1/2)+(\lambda/2\Delta)]} \exp(-x_0^2/2\Delta t)\left(\frac{1}{2\Delta t}\right)^{(3/2)+(\lambda/2\Delta)}. \tag{5.2}$$

For $2\Delta t \gg x_0^2$, the distribution $W(t)$ exhibits a power law behavior with the same exponent $(3/2)+(\lambda/2\Delta)$ as in the undamped case for $\gamma=0$. For weak noise this result is again in agreement with the estimate in Sec. III.

In the short time limit, $W(t)$ vanishes exponentially and shows a maximum about the finite-time singularity. In Fig. 3 we have depicted the first-passage-time distribution as a function of t. In Fig. 4 we illustrate the behavior of $W(t)$ in a log-log representation.

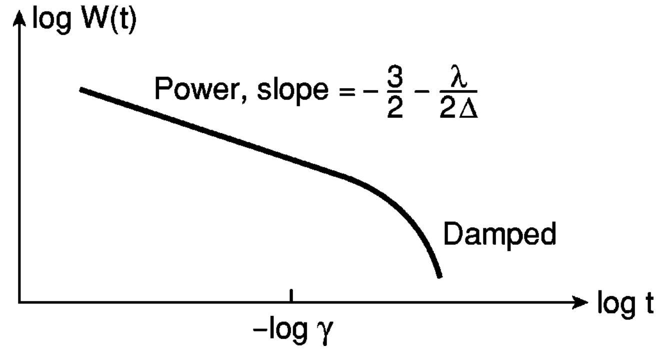

FIG. 4. In this figure we sketch the behavior of $W(t)$ in a log-log plot. At intermediate times earlier than $1/\gamma$ we have scaling behavior with exponent $(3/2)+(\lambda/2\Delta)$, corresponding to a constant negative slope. In the long time limit the curve dips down indicating the cross over to exponential behavior.

In this paper we have extended the model discussed in Ref. [15] to include a linear damping term. Not surprisingly, the damping changes the long time behavior of the physically relevant first-passage-time distribution. The finite-time singularity occurring at time t_0 in the noiseless case is still effectively resolved by the noise, becoming a random event, but the power law scaling behavior with scaling exponent $\alpha=(3/2)+(\lambda/2\Delta)$ is limited to early times compared with the cross-over time $1/\gamma$ set by the damping constant. In the long time limit beyond $1/\gamma$ the damping gives rise to an exponential falloff and the scaling property ceases to be valid. To the extent that the present simple model might apply to physical phenomena where damping is always present, we must conclude that an eventual power law scaling presumably is confined to a time window determined by the size of the damping.

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APPENDIX A: EXACT SOLUTION OF THE FOKKER-PLANCK EQUATION

In this appendix we discuss the exact solution of the Fokker-Planck equation in more detail. Denoting the normalized eigenfunction of H in Eq. (4.4) and the associated eigenvalues by Ψ_n and $\Delta^2 E_n/2$, respectively, we obtain, incorporating the initial condition $P(x,0)=\delta(x-x_0)$ and the gauge transformation, the following expression for the distribution:

$$P(x,t)=\sum_n e^{-\Delta E_n t/2} e^{-\gamma(x^2-x_0^2)/2\Delta}(x/x_0)^{-\lambda/2\Delta} \times \Psi_n(x)\Psi_n^*(x_0). \tag{A1}$$

By means of the transformation $\Psi(x)=x^{1+\lambda/2\Delta}\exp -\gamma x^2/2\Delta G(\gamma x^2/\Delta)$, it follows that G is a solution of the degenerate hypergeometric equation [39,40]. For the discrete spectrum we choose the polynomial form and further analysis shows that the eigenfunctions Ψ_n are given in terms of the Laguerre polynomials L_n^α [39,40]. For the normalized eigenfunctions we thus obtain

$$\Psi_n=\left[2\left(\frac{\gamma}{\Delta}\right)^{(3/2)+(\lambda/2\Delta)}\frac{\Gamma(n+1)}{\Gamma[(3/2)+(\lambda/2\Delta)+n]}\right]^{1/2}\times x^{1+\lambda/2\Delta}e^{-\gamma x^2/2\Delta}L_n^{(1/2)+(\lambda/2\Delta)}(\gamma x^2/\Delta), \quad (A2)$$

with discrete eigenvalue spectrum

$$E_n=4n\gamma/\Delta+2(\gamma/\Delta)(1+\lambda/\Delta). \quad (A3)$$

Inserting Eqs. (A2) and (A3) in Eq. (A1) and using the identity [39,40]

$$\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+\alpha+1)}z^nL_n^\alpha(x)L_n^\alpha(y)=\frac{(xyz)^{-\alpha/2}}{1-z}e^{-z(x+y)(1-z)}I_\alpha[2(xyz)^{1/2}/(1-z)], \quad (A4)$$

we finally obtain Eq. (4.11) for $P(x,t)$ and by the same analysis as in Ref. [15] the expression (4.12) for $W(t)$.

We note that for $\lambda\to 0$, using $I_{1/2}(x)=(2/\pi x)^{1/2}\sinh x$ [39], the expression (4.11) takes the form

$$P(x,t)=\left[\frac{\gamma}{\pi\Delta(1-e^{-2\gamma t})}\right]^{1/2}\left[\exp\left(-\frac{\gamma}{\Delta}\frac{(x-x_0e^{-\gamma t})^2}{1-e^{-2\gamma t}}\right)-\exp\left(-\frac{\gamma}{\Delta}\frac{(x+x_0e^{-\gamma t})^2}{1-e^{-2\gamma t}}\right)\right], \quad (A5)$$

i.e., the mirror case of the noise driven overdamped oscillator. Note that for $\Delta\to 0$, the variable x lies on the noiseless orbit $x\to x_0\exp(-\gamma t)$ and P vanishes for $x=0$.

APPENDIX B: SMALL NOISE LIMIT-SADDLE POINT ANALYSIS

In this appendix we perform for completion, a weak noise saddle point analysis of the exact expression in Eq. (4.11) along the same lines as in Ref. [15]. This analysis requires that we consider both large order and argument of the Bessel function $I_\nu(x)$. This is easily done by Laplace's method using a convenient spectral representation [39,40]

$$I_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)}\int_0^\pi \cosh(x\cos\theta)\sin^{2\nu}\theta d\theta. \quad (B1)$$

Inserting Eq. (B1) in Eq. (4.11) we have

$$P(x,t)=\frac{1}{4\pi\sqrt{2}}\frac{1}{\tilde{x}_0}\frac{\lambda}{\Delta}e^{\lambda/2\Delta}\left(\frac{\gamma\tilde{x}_0^2}{\lambda\sinh\gamma t}\right)^{(1/2)+(\lambda/2\Delta)}\times e^{-(\lambda/4\Delta)(\tilde{x}^2+\tilde{x}_0^2)/(\tilde{x}\tilde{x}_0\sinh u)}e^{\gamma t/2}\times\int_0^\pi d\theta\frac{\sin\theta}{\sinh u}(e^{(\lambda/\Delta)[\ln\sin\theta+(1/2)\cos\theta/\sinh u]}+e^{(\lambda/\Delta)[\ln\sin\theta-(1/2)\cos\theta/\sinh u]}), \quad (B2)$$

where u is defined by $\sinh u=\lambda\sinh\gamma t/(2\gamma\tilde{x}\tilde{x}_0)$, $\tilde{x}=x\exp(\gamma t/2)$, and $\tilde{x}_0=x_0\exp(-\gamma t/2)$. Setting $f_\pm(\theta)=\ln\sin\theta\pm(1/2)\cos\theta/\sinh u$ the saddle points for small Δ are given by $\cos\theta_\pm=\pm\exp(-u)$ for $x>0$ and $\cos\theta_\pm=\mp\exp(u)$ for $x<0$. For $x>0$ we have $f_+(\theta_+)=f_-(\theta_-)=(1/2)[\ln(1-e^{-2u})+e^u/\sinh u]$ and $f''_+(\theta_+)=f''_-(\theta_-)=-\coth u$, and we obtain the weak noise result for $x>0$

$$P(x,t)=\left(\frac{\lambda}{4\pi\Delta}\right)^{1/2}\frac{e^{\gamma t/2}}{\tilde{x}_0}\left(\frac{\gamma\tilde{x}_0^2(1-e^{-2u})}{\lambda\sinh\gamma t}\right)^{(1/2)+(\lambda/2\Delta)}\times\frac{e^{-(\lambda/2\Delta)(\tilde{x}^2+\tilde{x}_0^2-2\tilde{x}\tilde{x}_0\cosh u)/\tilde{x}\tilde{x}_0\sinh u}}{(\sinh u\cosh u)^{1/2}}. \quad (B3)$$

For $\Delta\to 0$ the factor $\tilde{x}^2+\tilde{x}_0^2-2\tilde{x}\tilde{x}_0\cosh u$ in the exponent in Eq. (B3) locks onto zero, thus setting $\tilde{x}^2+\tilde{x}_0^2-2\tilde{x}\tilde{x}_0\cosh u=0$ and inserting $\sinh u=\lambda\sinh(\gamma t)/2\gamma\tilde{x}\tilde{x}_0$ we obtain

$$\tilde{x}^2+\tilde{x}_0^2-[(2\tilde{x}\tilde{x}_0)^2+(\lambda\sinh\gamma t/\gamma)^2]^{1/2}=0. \quad (B4)$$

Finally, setting $\tilde{x}=x\exp(\gamma t/2)$ and $\tilde{x}_0=x_0\exp(-\gamma t/2)$ we obtain after some reduction

$$x(t)=\sqrt{x_0^2\exp(-2\gamma t)-(\lambda/2\gamma)[1-\exp(-2\gamma t)]}, \quad (B5)$$

which by simple inspection is equivalent to Eq. (2.3). For $\gamma\to 0$ we have $x=\sqrt{x_0^2-\lambda t}$; for $\lambda\to 0$, $x=x_0e^{-\gamma t}$.

[1] M.I. Freidlin and A.D. Wentzel, *Random Perturbations of Dynamical Systems*, 2nd ed. (Springer, New York, 1998).
[2] R.M. Kerr and A. Brandenburg, Phys. Rev. Lett. **83**, 1155 (1999).
[3] M.P. Brenner, J. Eggers, K. Joseph, S. Nagel, and X.D. Shi, Phys. Fluids **9**, 1573 (1997).
[4] M.P. Brenner, L. Levitov, and E.O. Budrene, Biophys. J. **74**, 1677 (1998).
[5] I. Cohen, M.P. Brenner, J. Eggers, and S.R. Nagel, Phys. Rev. Lett. **83**, 1147 (1999).
[6] P. Constantin, Proc. Natl. Acad. Sci. U.S.A. **94**, 12 761 (1997).
[7] J. Eggers, Rev. Mod. Phys. **69**, 865 (1997).
[8] M.P. Brenner, Nature (London) **403**, 377 (2000).
[9] D. Sornette and J.V. Andersen, Int. J. Mod. Phys. C **13**, 171 (2002).
[10] K. Ide and D. Sornette, Physica A **307**, 63 (2001).
[11] D. Sornette and A. Helmstetter, Phys. Rev. Lett. **89**, 158501 (2002).
[12] S. Gluzman, J. Andersen, and D. Sornette, J. Seismol. **32**, 122 (2001).

[13] A. Johansen and D. Sornette, Physica A **294**, 465 (2002).
[14] J. Eggers, Phys. Rev. Lett. **89**, 084502 (2002).
[15] H.C. Fogedby and V. Poukaradze, Phys. Rev. E **66**, 021103 (2002).
[16] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, 2001).
[17] A.J. Bray, Phys. Rev. E **62**, 103 (2000).
[18] J. Farago, Europhys. Lett. **52**, 379 (2000).
[19] F. Lillo, S. Micciché, and R.N. Mantegna, e-print cond-mat/0203442.
[20] R.L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963).
[21] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1989).
[22] S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, London, 1976).
[23] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1994).
[24] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).
[25] C.W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, New York, 1997).
[26] H.C. Fogedby, Phys. Rev. E **59**, 5065 (1999).
[27] H.C. Fogedby, Phys. Rev. E **60**, 4950 (1999).
[28] H.C. Fogedby and A. Brandenburg, Phys. Rev. E **66**, 016604 (2002).
[29] R.V. Roy, in *Computational Stochastic Mechanics*, edited by A.H.-D. Cheng and C.Y. Yang (Elsevier, Southampton, 1993).
[30] R. Graham, *Springer Tracts in Modern Physics* (Springer, Berlin, 1973), Vol. 66.
[31] R. Graham in *Noise in Nonlinear Dynamical Systems*, edited by F. Moss and P.E.V. McClintock, Theory of Continuous Fokker-Planck Systems Vol. 1 (Cambridge University Press, Cambridge, 1989).
[32] G. Falkovich, I. Kolokolov, V. Lebedev, and A. Migdal, Phys. Rev. E **54**, 4896 (1996).
[33] P.C. Martin, E.D. Siggia, and H.A. Rose, Phys. Rev. A **8**, 423 (1973).
[34] H.K. Janssen, Z. Phys. B **23**, 377 (1976).
[35] S.H. Strogatz, *Nonlinear Dynamics and Chaos* (Perseus Books, Reading, 1994).
[36] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).
[37] L. Landau and E. Lifshitz, *Mechanics* (Pergamon, Oxford, 1959).
[38] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Massachusetts, 1980).
[39] N.N. Lebedev, *Special Functions and Their Applications* (Dover, New York, 1972).
[40] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965).
[41] J. Mathews and R.L. Walker, *Mathematical Methods of Physics* (Benjamin, Menlo Park, 1973).